Randall

Calculus
Robots 101
Applied problem
Rubric

	Points	Earned
Part One		
4 a. Shows work correct answer	2	2
b. Shows work correct answer	2	2
c. Shows work correct answer	2	2
d. Shows work correct answer	2	2
5. Shows work correct answer	2	2
Part Two Find β		
6.a. Correct answer/shows work	4	4
b. Adds the squares of Equation 1 and 2/shows work	4	4
c. Simplifies equation	2	2
d. Finds sinβ/shows work Missing	2	0
e. correct answer/shows work	2	2
Part Three Find α		
7a. shows work/creates equation 4 and 5	4	4
b. Subs in values and simplifies	4	4
c. Correct answer/shows work uses crammer's rule properly solves for sin α and cosα	10	10
d. Solves for both α angles	4	4
Part Four: Exercise		
1. Draw a sketch using a protractor	4	4
Verify analytic measures to find (x_2, y_2)	4	4
Repeat with $\alpha = -45^o$ and $\beta = 30^o$	8	8
Repeat with $\alpha = 45^o$ and $\beta = -30^o$	8	8
TOTAL	70	68/70

97%

A

Pre-Calculus
Applied Problem
Robotics 101

Name Randall Wallace

Anyone who has watched a robot perform its tasks cannot help but be impressed by how it works. In this real world example we offer some insight into the mathematics that goes into designing such a robot.

In this problem the robot is a three-dimensional robot whose arms consist of upper and lower parts of length L_1 and L_2, respectively (see figure 1). The joint linking the two parts of that arm is called the elbow. The place where the robots arm meets his body we will call the origin. In this article, we assume that the robots arm moves only in a plane from front to back, and moves just as our arms do, except that the robot's upper and lower arm's, unlike ours, can rotate a full 360^0 in either direction (see figure 2). The robot's right arm is used to grasp objects and move them to another location in the plane of his arm's movements. Your job is to decide how to bend its upper and lower arms so that the robot can accomplish this task.

Part I

In determining how the robot can reach certain object (points):

1. You need to concern yourself with the angle called α that the upper arm makes with the horizontal. The angle that the lower arm makes in relation to the upper arm extended will be called angle β. (See figure 3).
2. When you measure angles, counterclockwise rotation will be considered positive and clockwise rotation will be considered negative.
3. Imagine that the tip of the upper arm moves to the point with coordinates (x_1,y_1) and that the tip of the lower arm moves to a point (x_2,y_2). (see figure 3) The object you want the robots hand to grasp is located at the point (x_2,y_2).
4. You first need to express (x_2,y_2) in terms of $\mathbf{L_1}$, $\mathbf{L_2}$, $\boldsymbol{\alpha}$, **and** $\boldsymbol{\beta}$. This can be done using the relationship shown in figure 4. To solve for x_2, you need to calculate it in steps.
   a. using figure 4, solve for $\cos\alpha$.
   b. Using the equation above from 4a., solve for x_1. Hold onto this equation to use as a substitution for x_1 in 4c.
   c. Now use trigonometry to solve for the distance between x_1 and x_2.
   d. Solve for x_2. We will call this equation (1).
5. There is a similar relationship for the y coordinates, only now we use the sine function. Solve for y_2. We will call this equation (2). Express the coordinates of the point (x_2,y_2) in terms of $\mathbf{L_1}$, $\mathbf{L_2}$, $\boldsymbol{\alpha}$, **and** $\boldsymbol{\beta}$.

Part II

Finding β

6. Because the lengths of the upper and lower arms are fixed, you need to determine the values of α and β to know how to move the robots arm.
   Taking a specific numerical example make the process of getting an equation easier. Using L_1=5 and L_2 = 3, have the robot pick up an object at (6,2). What should the angles α and β be?
   a. Make the substitutions into the equations (1) and (2) above in number 5.
   b. Add the squares of the two equations. Use the Pythagorean identities to simplify the results. You will come up with an answer in terms of α and β. Call this equation (3).
   c. Now use the sum formula for cosine and simplify the equation.
   d. Use a reference angle to find sinβ. You will use this later.
   e. Remember that the domain for β is, $[0°, 360°]$. Use the equation in 6c to solve for β. There will be two answers but only use the principal angle. See figure 5.

Part III

Finding α

7. a. Now that you have β, you need to find α by using trigonometric identities again. This time use the sum identities. Sub the identities into the equation (1) and (2). We will call these equations ~~(3)~~ 4 and ~~(4)~~ 5.
   b. Take those values and sub them into your equation ~~(3)~~ 4 and ~~(4)~~ 5, using the values from #6. Again simplify the equations. We will call these equations (6) and (7).
   c. Now that you have the two equations using the data from 7b., use Crammers rule to solve for Sinα and cosα.
   It will be easier to calculate if you do the following.
   Let cosα = x and sinα = y
   Rewrite equation (6) and (7) as using x and y. Crammers Rule uses matrices. Then use your calculator to figure the Determinate (D).
   Crammer's Rule: $x = D_x \div D$, $y = D_y \div D$
   d. Then solve for α. Also use the negative answer because that is less movement for the robot. Remember that the concept of the signs of the trigonometric functions in the various quadrants plays a big role in determining how we move the robot arm.

Part IV

Exercise

1. Let L_2=3 and L_1=5 α=45°, and β=30°. Draw a sketch and use equation (1) and (2) to find the coordinates (x_2, y_2). Verify that your sketch and your analytic results are the same. Do not forget to measure β from the extension of the upper arm. Repeat, using first α = -45° and β = 30° and then α =45° and β = -30° and leaving L_1 and L_2 as before.

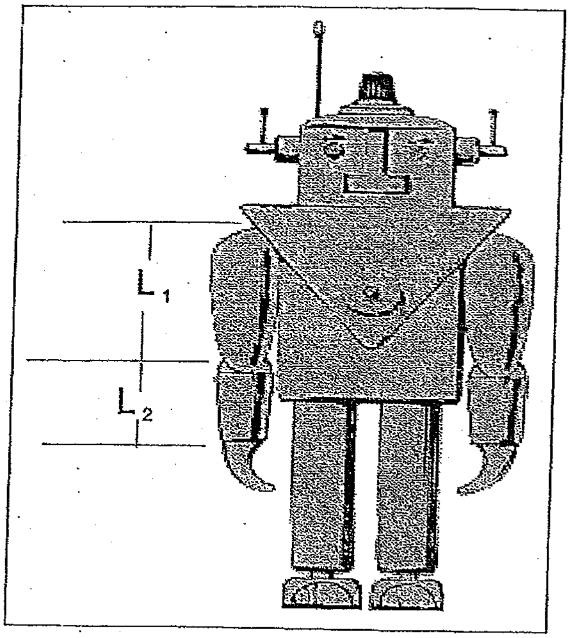


Fig. 1 Each of the two parts of Robby's right arm can rotate a full 360°.

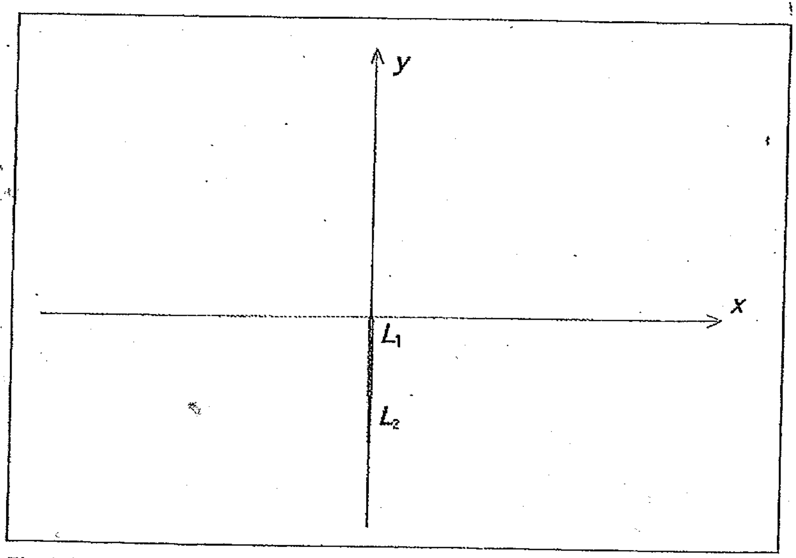


Fig. 2 The schematic represents a side view of Robby's arm.

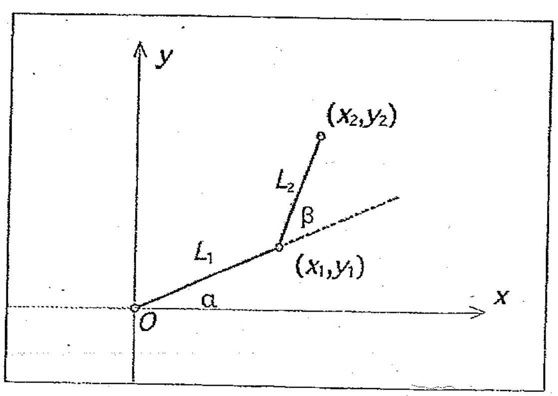


Fig. 3 When L_1 and L_2 rotate through angles α and β, respectively, we are able to locate the points with coordinates (x_1, y_1) and (x_2, y_2).

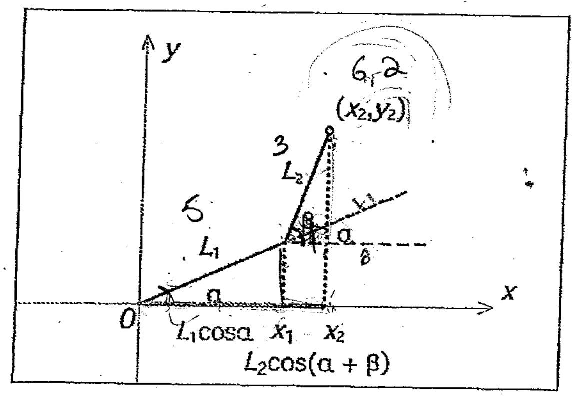


Fig. 4 Trigonometry provides a means to express the coordinates (x_2, y_2).

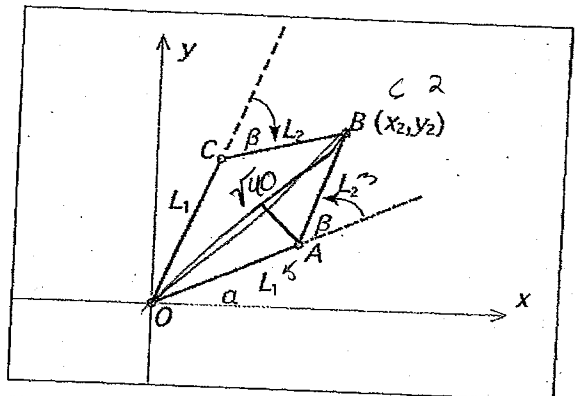


Fig. 5 Each of the two values for β produces a different way to get to point B.

SOH CAH TOA

4.) $L_2 \cos(\alpha+\beta) = x$
$L_2 \sin(\alpha+\beta) = y$

$\left(L_2\cos(\alpha+\beta), L_2\sin(\alpha+\beta)\right)$

a.)
$\cos\alpha = \frac{x_1}{L_1}$

b.) $L_1\cos\alpha = x_1$

c.) $L_2\cos(\alpha+\beta) =$

d.)
$L_2\cos(\alpha+\beta) + L_1\cos\alpha = x_2$ - Equation 1

5.) $L_1\sin\alpha + L_2\sin(\alpha+\beta) = y^2$ - Equation 2

$\left(L_2\cos(\alpha+\beta) + L_1\cos\alpha, L_1\sin\alpha + L_2\sin(\alpha+\beta)\right)$

A) $x_2 = L_2(\cos(\alpha+\beta) + \cos\alpha \cdot L_1$

$6 = 3(\cos(\alpha+\beta) + \cos\alpha \cdot 5$

$2 = 3(\sin(\alpha+\beta) + \sin\alpha \cdot 5$

B.) $6^2 = 3\cos(\alpha+\beta) + 5\cos\alpha^{\,2}$ $\qquad 2^2 = 3\sin(\alpha+\beta) + 5\sin\alpha^{\,2}$

$36 = 9\cos^2(\alpha+\beta) + 30\cos(\alpha+\beta \cdot \cos\alpha + 25\cos^2\alpha$

$4 = 9\sin^2(\alpha+\beta) + 15\sin(\alpha+\beta) \cdot \sin\alpha) + 15(\sin\alpha+\beta) \cdot \sin(\alpha) + 25\sin^2\alpha$

$40 = 9\cos^2(\alpha+\beta) + 30\cos(\alpha+\beta) \cdot \cos(\alpha) + 25\cos^2\alpha + 9\sin^2(\alpha+b) + 30\sin(\alpha+\beta)$
$\cdot \sin(\alpha) + 25\sin^2\alpha$

$40 = 30\cos(\alpha+\beta) \cdot \cos\alpha + 30\sin(\alpha+\beta) \cdot \sin\alpha + 9\cos^2(\alpha+\beta) + \sin^2(\alpha+\beta) + 2\sin^2\alpha +$
$\cos^2\alpha$

$40 = \frac{30\cos(\alpha+\beta)}{30} \cdot \cos\alpha + \frac{30}{30}\sin(\alpha+\beta)\sin\alpha + 9 + 25$
$\frac{6}{30}$ $\qquad -36$

$\frac{1}{5} = \cos^2\alpha\cos\beta - \cancel{(\sin\alpha\sin\beta\cos\alpha)} + \sin^2\alpha\cos\beta + \cancel{(\cos\alpha\sin\beta\sin\alpha)}$

$\frac{1}{5} = \cos^2\alpha\cos\beta + \sin^2\alpha\cos\beta$

c.) $\frac{1}{5} = \cos\beta(1)$ - Equation 3

D.) $\cos\beta = \frac{1}{5}$

Part II

(a) $L_1 = 5$ $\quad\quad\quad\quad$ $(6, 2)$

$L_2 = 3$

$\boxed{(L_2 \cos(\alpha + \beta),\ L_1 \sin\alpha + L_2 \sin(\alpha + \beta))}$

$6 \quad\quad\quad 2$

$5\sin\alpha + 3\sin(\alpha + \beta) = 2$

$L_2 \cos(\alpha + \beta) = 6$

$\frac{3\cos(\alpha + \beta)}{3} = \frac{6}{3}$ $\quad\quad\quad$ $\sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = 2$

$\alpha + \beta = \cos^{-1}(2)$

c.) $\cos^{-1}\left(\frac{1}{5}\right) = \beta$

$78.5^\circ = \beta$

$\sin\beta = \frac{\sqrt{24}}{5}$

$$\boxed{\sin\beta = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}}$$

Part III

7.) a.) $L_2\cos(\alpha+\beta) + L_1\cos\alpha = x_0$ $\qquad$ $L_1\sin\alpha + L_2\sin(\alpha+\beta) = y^2$

eq 4 $\quad 3(\cos(\alpha+78.5^\circ)) + 5\cos\alpha = 6$ $\qquad$ eq 5 $\quad 5\sin\alpha + 3\sin(\alpha+78.5^\circ) = 2$

$6 = 3(\cos\alpha\cos\beta - \sin\alpha\cos\beta) + 5\cos\alpha$ $\qquad$ $2 = 5\sin\alpha + 3(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$

$3(\cos\alpha\,\frac{1}{5} - \sin\alpha\,\frac{1}{5}) + 5\cos\alpha$ $\qquad$ $2 = 5\sin\alpha + 3(\sin\alpha\,\frac{1}{5} + \frac{\sqrt{24}}{5}\cos\alpha)$

b.) $\frac{3}{5}\cos\alpha - \frac{3}{5}\sin\alpha + 5\cos\alpha$

$5\sin\alpha + \frac{3}{5}\sin\alpha + \frac{6\sqrt{6}}{5}\cos\alpha$

$6 = \frac{28}{25}\cos\alpha\ \frac{3}{5}\sin\alpha$ Eq 6 $\qquad$ $2 = \frac{28}{5}\sin\alpha + \frac{6\sqrt{6}}{5}\cos\alpha$ Eq 7

$D = ay - by$ $\qquad$ $cy \quad dx$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\qquad$ $\frac{28}{5} \quad \frac{3}{5}$ $\qquad$ $\frac{6\sqrt{6}}{5} \quad \frac{28}{5}$

$\frac{28}{5}\cdot\frac{28}{5} - \frac{6\sqrt{6}}{5}\cdot\frac{6\sqrt{6}}{5} = 40$

$D_x = \begin{vmatrix} s & d \\ t & b \end{vmatrix} = sd - bt = \begin{vmatrix} 6 & \frac{6\sqrt{6}}{5} \\ 2 & \frac{28}{5} \end{vmatrix}$

$\frac{6\cdot 28}{5} - \frac{2\cdot 6\sqrt{6}}{5} = 39.5$

$D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$ $\quad at - sc$

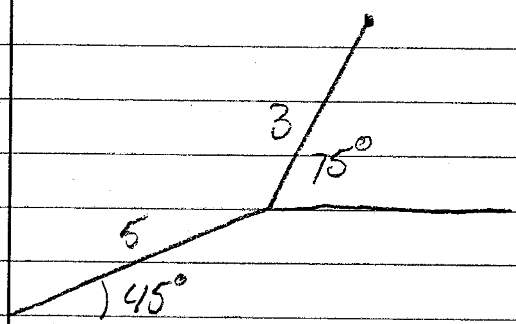
$$\begin{vmatrix} \frac{28}{5} & 6 \\ \frac{6\sqrt{6}}{5} & 2 \end{vmatrix}$$

$\frac{28}{5}\cdot 2 - \frac{6\cdot 6\sqrt{6}}{5} = 6.4$

d.) $a\cos^{-1}(0.99) = 9.07$

$a\sin^{-1}(-0.16) = -9.25$

Part IV



$L_1 \cos(\alpha + \beta) + L_2 \cos\alpha = x_2$

$5\cos(45+30) + 3\cos 45$

(75)

$x_2 = 3.4$

$L_1 \sin\alpha + L_2 \sin(\alpha+\beta)$

$3\sin 45 + 5\sin(75)$

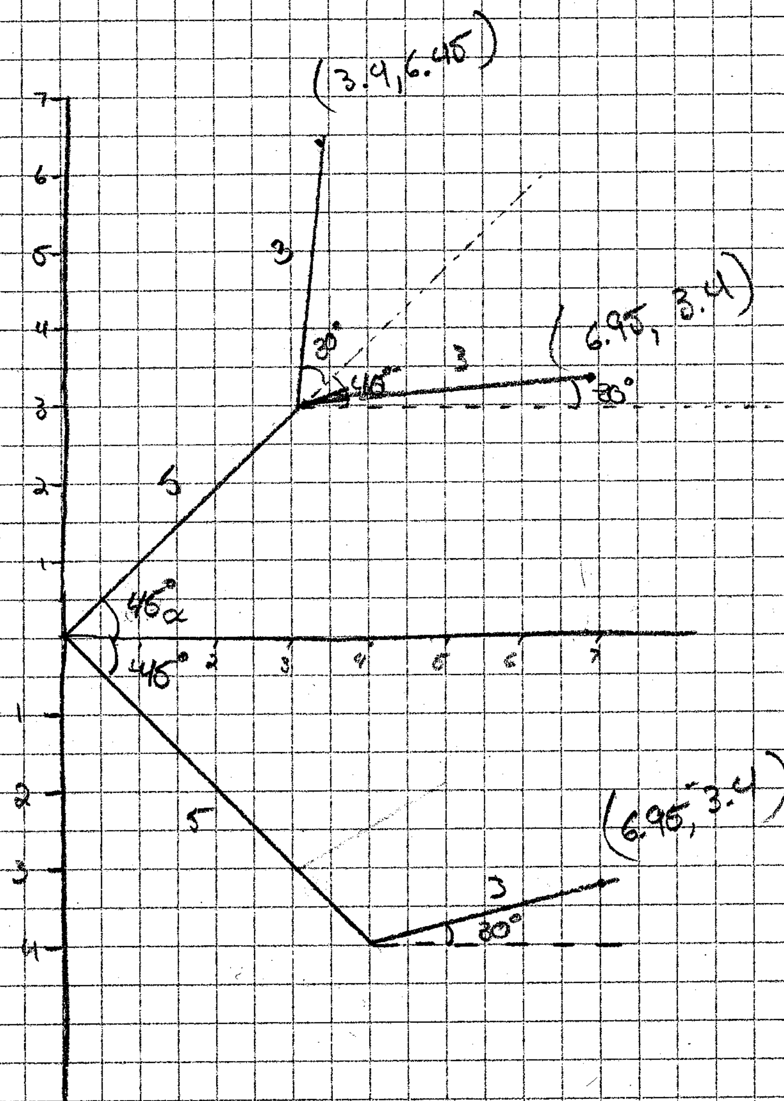
$y_2 = 6.95$

$5\sin(-45+30) + \sin(-45)\cdot 3 = -3.4$

$5\sin(45-30) + \sin(45)\cdot 3 = 3.4$

$5\cos(-45+30) + \cos(-45)\cdot 3 = 6.95$

$5\cos(45-30) + \cos(45)\cdot 3 = 6.95$



(3.9, 6.45)
(6.95, 3.4)
(6.95, 3.4)
3
30°
45°
3
30°
5
45°
α
45°
5
3
30°